Astron. Astrophys. 333, L79–L82 (1998)

ASTRONOMY AND ASTROPHYSICS

LETTER

Letter to the Editor

A robust and efficient method for calculating the magnification of extended sources caused by gravitational lenses

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Received 16 March 1998 / Accepted 30 March 1998

Abstract. To determine the magnification of an extended source caused by gravitational lensing one has to perform a two-dimensional integral over point-source magnifications in general. Since the point-source magnification jumps to an infinite value on caustics, special care is required. For a uniformly bright source, it has been shown earlier that the calculation simplifies if one determines the magnification from the area of the images of the extended source by applying Green's theorem so that one ends up with a one-dimensional integration over the image boundaries. This approach is discussed here in detail, and it is shown that it can be used to yield a robust and efficient method also for limb-darkened sources. It is also shown that the centroid shift can be calculated in a similar way.

Key words: methods: numerical – gravitational lensing – planetary systems – binaries: general – dark matter

1. Introduction

For fitting light curves for the ongoing microlensing events, there is a need for robust and efficient methods for calculating the magnification of extended sources, which are not limited to point-lenses. Among the observed events, the presence of binary lenses is a reality (Dominik & Hirshfeld 1994,1996; Udalski et al. 1994; Alard et al. 1995; Bennett et al. 1996), and planetary events involve a special case of a binary lens. In addition, for some configurations, the light curve for a limb-darkened source will differ significantly from that of a uniformly bright source. The limb-darkening effect has recently been observed in the galactic microlensing event MACHO 97-BLG-28, which involves both an extended source and a binary lens, by the PLANET collaboration (Albrow et al. 1998a,b); the fitting has been done by myself using the algorithm described in this letter.

If one wants to integrate the point-source magnification in two dimensions one has to take special care of the position of the caustics, where the point-source magnification becomes infinite. While this integration can be performed easily for a point-mass lens (e.g. Schneider et al. 1992, p. 313; Witt & Mao 1994; Sahu 1994; Dominik 1996), this would be a difficult task for a general lens (e.g. a binary lens), especially at a cusp singularity. In contrast, the area of the images of the extended source and therefore its magnification remains continuous when the source hits a caustic. The determination of the extended source magnification from the boundaries of the image areas has been used to analyze the images of background galaxies behind a cluster of galaxies (Dominik 1993). The image boundaries can be obtained with a contour plot of an implicit function describing the source boundary in the lens plane (Schramm & Kayser 1987). This method has been expanded with routines for correcting, testing and finally analyzing the contour line in order to produce an efficient and safe algorithm (Dominik 1995). In that paper, it is noted that it is easy to analyze the images from the contour line data, and an example is given, where quantities such as the area, width, length and curvature of the image have been determined.

Concerning microlensing light curves, it has been noted by Bennett & Rhie (1996) that it is advantageous to integrate in the lens plane rather than in the source plane to determine the magnification of an extended source. For uniformly bright sources, Gould & Gaucherel (1997) proposed applying Green's theorem so that only one integration along the image boundary must be performed rather than two over the image area. This approach is identical to that used earlier (Dominik 1993, 1995). The contour plot method is the most convenient way to obtain data points on the image boundary from which the area can be calculated. In Sect. 2, this general approach is described, Sect. 3 gives details for a uniformly bright source, and Sect. 4 shows how this approach can also be used for limb-darkened sources, in which case an easy-to-perform two-dimensional integration remains. In Sect. 5, the calculation of the centroid shift is discussed.

2. Magnification and Green's theorem

The magnification is given by the ratio of the area of the images to the area of the source in the absence of the lens. Let (y_1, y_2)

⋆ Work carried out at the Space Telescope Science Institute is financed by a research grant from Deutsche Forschungsgemeinschaft

denote arbitrary cartesian source coordinates and (x_1, x_2) denote corresponding image coordinates, where in the absence of the lens $\boldsymbol{y} = \boldsymbol{x}$. Let $I_\mathrm{s}(\boldsymbol{y})$ denote the surface brightness of the source and $I_\mathrm{l}(\boldsymbol{x})$ denote the surface brightness of the images. The conservation of photon number then requires (see e.g. Schneider et al. 1992, p. 33; Schramm & Kayser 1987; Kayser & Schramm 1988) that

$$I_\mathrm{l}(\boldsymbol{x}) = I_\mathrm{s}(\boldsymbol{y}(\boldsymbol{x}))\,, \tag{1}$$

where $\boldsymbol{y}(\boldsymbol{x})$ gives the source position $\boldsymbol{y}$ related to the image position $\boldsymbol{x}$.

With A_s being the region in the (y_1, y_2)-plane subtended by the source and A_l being the region in the (x_1, x_2)-plane subtended by the images, the magnification is given by

$$\mu = \frac{\int_{A_\mathrm{l}} I_\mathrm{l}(\boldsymbol{x})\,\mathrm{d}x_1\,\mathrm{d}x_2}{\int_{A_\mathrm{s}} I_\mathrm{s}(\boldsymbol{y})\,\mathrm{d}y_1\,\mathrm{d}y_2}\,. \tag{2}$$

In the limit of a point source, one obtains $\mu = \sum \tilde{\mu}(\boldsymbol{x}_j)$, where the sum runs over the m images $\boldsymbol{x}_j$ of the source, and $\tilde{\mu}$ is given by

$$\tilde{\mu}(\boldsymbol{x}_j) = \frac{1}{\left|\det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}\right)(\boldsymbol{x}_j)\right|}\,. \tag{3}$$

The normalization of the brightness profile function I_s can be chosen so that the integral in the denominator (Eq. (2)) becomes the area of the source, and for a circular source of radius R_src one obtains

$$\int_{A_\mathrm{s}} I_\mathrm{s}(\boldsymbol{y})\,\mathrm{d}y_1\,\mathrm{d}y_2 = \pi R_\mathrm{src}^2\,. \tag{4}$$

For a uniformly bright source, one has $I_\mathrm{s}(\boldsymbol{y}) = 1$ for positions within the source, so that $I_\mathrm{l}(\boldsymbol{x}) = 1$ for positions within the images, and the magnification becomes

$$\mu = \frac{1}{\pi R_\mathrm{src}^2} \int_{A_\mathrm{l}} \mathrm{d}x_1\,\mathrm{d}x_2\,, \tag{5}$$

where the remaining integral is just the area of the images.

Green's theorem now states that for two functions $P(x_1, x_2)$ and $Q(x_1, x_2)$ which are continous in a region A and whose partial derivatives $\partial P/\partial x_2$ and $\partial Q/\partial x_1$ are also continuous in A,

$$\int_A \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right) \mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\partial A} P\,\mathrm{d}x_1 + Q\,\mathrm{d}x_2\,, \tag{6}$$

where ∂A denotes the boundary of the region A, which has to be piecewise-smooth. Note that the values of P and Q outside the region A do not play a role and can be chosen arbitrarily as long as P and Q satisfy the conditions above.

3. Uniformly bright sources

For a uniformly bright source, one can now choose $P = -\frac{1}{2}x_2$ and $Q = \frac{1}{2}x_1$ to obtain

$$\mu = \frac{1}{2\pi R_\mathrm{src}^2} \int_{\partial A} x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1\,. \tag{7}$$

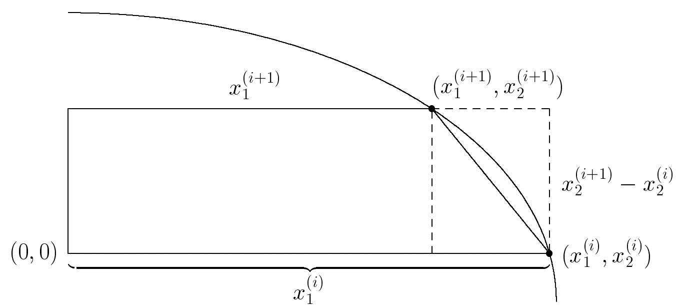

Fig. 1. Two successive points $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(i+1)}$ on the image boundary and the corresponding area segment

For n discrete positions $\boldsymbol{x}^{(i)}$ on the image boundary, where $\boldsymbol{x}^{(n+i)} = \boldsymbol{x}^{(i)}$, which are obtained e.g. by a contour plot, the magnification can be approximated by

$$\mu = \frac{1}{2\pi R_\mathrm{src}^2} \sum_{i=1}^{n} \left[x_1^{(i)}\,(x_2^{(i+1)} - x_2^{(i)}) - x_2^{(i)}\,(x_1^{(i+1)} - x_1^{(i)})\right]\,. \tag{8}$$

This means that one adds up rectangular segments with the length $x_1^{(i)}$ and the width $\Delta x_2^{(i)} = x_2^{(i+1)} - x_2^{(i)}$ (and corresponding segments by interchanging the axes). However $x_1^{(i+1)}$ has an equal "right" to be convolved with $\Delta x_2(i)$, yielding the symmetric version

$$\mu = \frac{1}{4\pi R_\mathrm{src}^2} \sum_{i=1}^{n} \left[(x_1^{(i)} + x_1^{(i+1)})\,(x_2^{(i+1)} - x_2^{(i)}) - (x_2^{(i)} + x_2^{(i+1)})\,(x_1^{(i+1)} - x_1^{(i)})\right]\,, \tag{9}$$

which sums over trapezoidal segments having an area which is the mean of the rectangular areas formed from segments with length $x_1^{(i)}$ and $x_1^{(i+1)}$, and corresponds to a replacement of the true boundary by a polygon with n corners at $\boldsymbol{x}^{(i)}$ (see also Fig. 1). By increasing the number of points used, the magnification can be determined to the desired precision.

4. Limb-darkened sources

Let us now consider a limb-darkened source at $\boldsymbol{y}^{(0)}$ with radius R_src, with the brightness profile

$$I_\mathrm{s}(\boldsymbol{y}) = f_\mathrm{I}\left(\frac{\left(\boldsymbol{y} - \boldsymbol{y}^{(0)}\right)^2}{R_\mathrm{src}^2}\,;\,\tilde{u}\right)\,, \tag{10}$$

where

$$f_\mathrm{I}(r;\tilde{u}) = \frac{1}{1 - \tilde{u}/3}\,\left(\tilde{u}\,\sqrt{1 - r^2} + 1 - \tilde{u}\right) \tag{11}$$

for $0 \le r \le 1$, and $f_\mathrm{I}(r,\tilde{u}) = 0$ for $r > 1$. This profile is normalized so that Eq. (4) is satisfied. The parameter $\tilde{u}$ is chosen from the interval $[0, 1]$, where $\tilde{u} = 0$ corresponds to a uniformly bright source. This brightness profile is the simple 'linear' model[1] widely used, though for some type of stars, one should add other terms (e.g. Claret et al. 1995). The general ideas of the approach discussed here do not depend on this special choice, especially terms like $(1-r^2)^n$ (i.e. like $\cos^{2n}\theta$) can be treated in complete analogy to the $\sqrt{1-r^2}$-term.[2]

From Eqs. (2) and (4), one obtains the magnification as

$$\mu = \frac{1}{\pi R_{\rm src}^2} \int_{A_{\rm l}} I_{\rm l}(\boldsymbol{x})\,{\rm d}x_1\,{\rm d}x_2\,, \tag{12}$$

To apply Green's theorem one has to find functions P and Q which satisfy

$$\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} = I_{\rm l}(x_1,x_2) = I_{\rm s}(\boldsymbol{y}(x_1,x_2))\,. \tag{13}$$

Such functions are given by

$$P(x_1,x_2) \;=\; -\frac{1}{2}\int_{x_2^{(0)}}^{x_2} I_{\rm s}\left(\boldsymbol{y}(x_1,x_2')\right)\,{\rm d}x_2'\,, \tag{14}$$

$$Q(x_1,x_2) \;=\; \frac{1}{2}\int_{x_1^{(0)}}^{x_1} I_{\rm s}\left(\boldsymbol{y}(x_1',x_2)\right)\,{\rm d}x_1'\,. \tag{15}$$

Since the brightness profile $I_{\rm s}$ and the lens equation $\boldsymbol{y}(\boldsymbol{x})$ are given in analytical form, the integrals can be evaluated numerically in general.

Since the brightness profile has an infinite slope at the limb of the source, it is advantageous to use another continuation for $r > 1$.[3] Let us write the brightness profile in the form

$$f_{\rm I}(r;\tilde{u}) = C(\tilde{u})\,(\tilde{u}\,B(r) + 1 - 2\tilde{u})\,, \tag{16}$$

where

$$C(\tilde{u}) = \frac{1}{1-\tilde{u}/3} \tag{17}$$

and

$$B(r) = \begin{cases} 1+\sqrt{1-r^2} & \text{for } 0 \le r < 1 \\ 1 & \text{for } r = 1 \\ 1-\sqrt{1-1/r^2} & \text{for } r > 1 \end{cases}\,. \tag{18}$$

This definition is identical to the previous one except for $r > 1$.[4]

The functions P and Q read

$$P(x_1,x_2) \;=\; -C/2 \left[(1-2\tilde{u})\,(x_2 - x_2^{(0)}) \;+ \right. \left. +\;\tilde{u}\int_{x_2^{(0)}}^{x_2} B(r(x_1,x_2'))\,{\rm d}x_2' \right]\,, \tag{19}$$

$$Q(x_1,x_2) \;=\; C/2 \left[(1-2\tilde{u})\,(x_1 - x_1^{(0)}) \;+ \right. \left. +\;\tilde{u}\int_{x_1^{(0)}}^{x_1} B(r(x_1',x_2))\,{\rm d}x_1' \right]\,, \tag{20}$$

where

$$r(x_1,x_2) = \frac{\left(\boldsymbol{y}(x_1,x_2) - \boldsymbol{y}^{(0)}\right)^2}{R_{\rm src}^2}\,. \tag{21}$$

The function $B(r)$ has been chosen, so that for all r, $B(r) > 0$ in order to avoid contributions of different sign in the numerical integration process. The tail of the function is limited to

$$\int_1^\infty B(r)\,dr = \frac{\pi}{2} - 1 \approx 0.571\,, \tag{22}$$

so that the integral is not dominated by the tail contribution.

One remaining point of interest is the choice of the lower integration bound $\boldsymbol{x}^{(0)}$. To avoid unnecessary integration in the tail region of $B(r)$ that would result for a fixed choice for $\boldsymbol{x}^{(0)}$ such as $(0,0)$, the lower integration bound can be chosen as the center of each image which can be determined as shown in the next section.

With these funtions P and Q one can approximate the magnification by the expression

$$\begin{aligned}\mu = \frac{1}{2\pi R_{\rm src}^2} \sum_{i=1}^{n} \Bigg\{ \Bigg[& Q\left(x_1^{(i)}, \frac{1}{2}(x_2^{(i)} + x_2^{(i+1)})\right) + \\ & + Q\left(x_1^{(i+1)}, \frac{1}{2}(x_2^{(i)} + x_2^{(i+1)})\right)\Bigg]\,(x_2^{(i+1)} - x_2^{(i)}) + \\ & + \Bigg[P\left(\frac{1}{2}(x_1^{(i)} + x_1^{(i+1)}), x_2^{(i)}\right) + \\ & + P\left(\frac{1}{2}(x_1^{(i)} + x_1^{(i+1)}), x_2^{(i+1)}\right)\Bigg]\,(x_1^{(i+1)} - x_1^{(i)})\Bigg\}\,, \end{aligned} \tag{23}$$

which uses the symmetries and reduces to Eq. (9) with $\boldsymbol{x}^{(0)} = (0,0)$ for the case $\tilde{u} = 0$ (uniformly bright source). An example for a light curve for a limb-darkened source and one with a uniformly bright source behind a binary lens is shown in Fig. 6.

5. The centroid shift

The position of the centroid of light can be determined in a similar way. For m images of the source, the position of the centroid of light $\boldsymbol{x}_{\rm c}$ is given by

$$\boldsymbol{x}_{\rm c} = \frac{\sum\limits_{j=1}^{m} \int_{A_{\rm j}} \boldsymbol{x} I_{\rm l}(\boldsymbol{x})\,{\rm d}x_1\,{\rm d}x_2}{\sum\limits_{j=1}^{m} \int_{A_{\rm j}} I_{\rm l}(\boldsymbol{x})\,{\rm d}x_1\,{\rm d}x_2} \tag{24}$$

where an integration has to be performed over the regions subtended by the images $A_{\rm j}$. The integral in the denominator is just that discussed in the previous section and the integral in the numerator differs only by an additional factor of $\boldsymbol{x}$. The integral can be solved in the way described in the last section by only replacing $I_{\rm l}(\boldsymbol{x})$ by $x_i\,I_{\rm l}(\boldsymbol{x})$ $(i = 1,2)$. For a uniformly bright source, the corresponding functions P and Q are

$$P(x_1,x_2) = -\frac{1}{2}x_1 x_2\,, \quad Q(x_1,x_2) = \frac{1}{4}x_1^2 \tag{25}$$

[1] The profile involves a linear term of $\cos\theta$, where θ is the angle between the normal to the surface and the direction to the observer.

[2] In fact, for the discussion of the MACHO 97-BLG-28 event (Albrow et al. 1998a,b), a $\sqrt[4]{1-r^2}$-term has also been included in the brightness profile.

[3] As mentioned earlier, the value for $r > 1$ does not play a role.

[4] One may also use other continuations for $r > 1$ which decrease differently for $r \to \infty$.

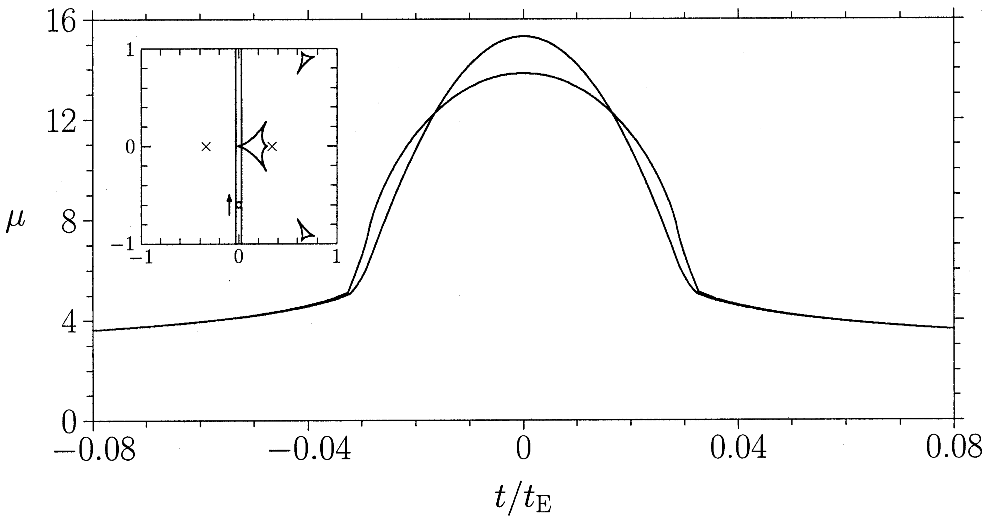

Fig. 2. The light magnification for a limb-darkened source with $\tilde{u} = 1$ (curve with larger peak amplification) and a uniformly bright source behind a binary lens. The angular source radius corresponds to 0.03 θ_{E}, where θ_{E} denotes the angular Einstein radius[5]. The mass ratio between the two lens components is 4, their angular separation is 0.68 θ_{E}. The source center passes perpendicularly to the line connecting the lens objects at a minimal impact of 0.004 θ_{E} from the midpoint towards the lighter component. t_{E} is the time in which the angular source position relative to the lens changes by θ_{E}, $t = 0$ corresponds to the point of time when the center of the source crossing the line connecting the lens components. The inset shows the caustics, the position of the lens objects (crosses), and the size and trajectory of the source which sweeps over a cusp caustic; the coordinates are multiples of θ_{E}.

for the x_1-component and

$$P(x_1, x_2) = -\frac{1}{4} x_2^2\,, \quad Q(x_1, x_2) = \frac{1}{2} x_1 x_2 \tag{26}$$

for the x_2-component, so that one obtains with Eq. (23) the expressions

$$\begin{aligned} x_{\mathrm{c},1} \;=\; & \frac{1}{8A} \sum_{i=1}^{n} \Big\{ ({x_1^{(i)}}^2 + {x_1^{(i+1)}}^2)\,(x_2^{(i+1)} - x_2^{(i)}) \;+ \\ & +\; ({x_1^{(i)}}^2 - {x_1^{(i+1)}}^2)(x_2^{(i+1)} + x_2^{(i)}) \Big\}\,, \end{aligned} \tag{27}$$

$$\begin{aligned} x_{\mathrm{c},2} \;=\; & -\frac{1}{8A} \sum_{i=1}^{n} \Big\{ ({x_2^{(i)}}^2 - {x_2^{(i+1)}}^2)\,(x_1^{(i+1)} + x_1^{(i)}) \;+ \\ & +\; ({x_2^{(i)}}^2 + {x_2^{(i+1)}}^2)(x_1^{(i+1)} - x_1^{(i)}) \Big\}\,, \end{aligned} \tag{28}$$

where A denotes the area of the image, which can be determined as shown in Sect. 3.

For a point source, the centroid's position can be written as

$$\boldsymbol{x}_{\mathrm{c}} = \frac{\sum_{j=1}^{m} \tilde{\mu}(\boldsymbol{x}_j)\, \boldsymbol{x}_j}{\sum_{j=1}^{m} \tilde{\mu}(\boldsymbol{x}_j)}\,. \tag{29}$$

If one compares this expression with that given by Eq. (24), one sees that the (finite) source size has cancelled out in Eq. (24).

Acknowledgements. I would like to thank P. Sackett and K. Sahu for reading and commenting on this manuscript.

[5] For a lens at a distance D_{d} and a source at a distance D_{s} from the observer, the angular Einstein radius for a lens of total mass M is given by $\theta_{\mathrm{E}} = \left(4GMc^{-2}D_{\mathrm{d}}^{-1}D_{\mathrm{s}}^{-1}(D_{\mathrm{s}} - D_{\mathrm{d}})\right)^{1/2}$.

References

Alard C., Mao S., Guibert J., 1995, A&A 300, L17

Albrow M., Beaulieu J.-P., Caldwell J., et al. (The PLANET collaboration), 1998a, 1997 PLANET monitoring of anomalous events — First detection of limb-darkening via microlensing, to appear in "Proceedings of the 4th international workshop on gravitational microlensing surveys" held in Paris, France, Jan 15-17

Albrow M., Beaulieu J.-P., Caldwell J., et al. (The PLANET collaboration), 1998b, in preparation

Bennett D. P., Alcock C., Allsman R. A., et al. (The MACHO collaboration), 1996, Nucl. Phys. Proc. Suppl. 51B, 152

Bennett D. P., Rhie S. H., 1996, ApJ 472, 660

Claret A., Díaz-Cordovés J., Gimenéz A., 1995, A&AS 114, 247

Dominik M., 1993, Effiziente Methoden zur Invertierung der Gravitationslinsengleichung und zur Analyse von Bildern ausgedehnter Quellen, Diploma thesis, Universität Dortmund

Dominik M., 1995, A&AS 109, 507

Dominik M., 1996, Galactic microlensing beyond the standard model, PhD thesis, Universität Dortmund

Dominik M., Hirshfeld A. C., 1994, A&A 289, L31

Dominik M., Hirshfeld A. C., 1996, A&A 313, 841

Gould A., Gaucherel C., ApJ 477, 580

Kayser R., Schramm T., 1988, A&A 191, 39

Sahu K., 1994, PASP 106, 942

Schneider P., Ehlers J., Falco E. E., 1992, Gravitational Lenses, Springer, Berlin

Schramm T., Kayser R., 1987, A&A 174, 361

Udalski A., Szymański M., Mao S., et al., 1994, ApJ 436, L103

Witt H. J., Mao S., 1994, ApJ 430, 505