Rachunkowe podstawy kosmologii

- Metryka jednorodnej i izotropowej czasoprzestrzeni
- Krzywizna przestrzeni
- Równania Einsteina
- Równania Friedmana
- Prawo Hubble'a
- Przesunięcie ku czerwieni podstawowym parametrem
- Wielkości obserwowane

Paradygmaty

- Izotropowy
- Jednorodny
- Ekspandujący homologicznie (<==>pr. Hubble'a)
- OTW słuszna ==> Jedna z 3 możliwych globalnych geometrii

Metric

$$ds^2 = c^2 dt^2 - dl^2$$

$$\begin{aligned} dl^2 &= a^2(t)\left(d\chi^2 + S^2(\chi)\left(d\theta^2 + \sin^2\theta\phi^2\right)\right) \\ S(\chi) &= \sin\chi \quad \text{if } k=+1, \qquad \chi \quad \text{if } k=0, \qquad \sinh\chi \quad \text{if } k=-1 \end{aligned}$$

$$g_{tt} = c^2 \qquad g_{\chi\chi} = -a^2(t) \qquad g_{\theta\theta} = -a^2(t)S^2(\chi) \qquad g_{\phi\phi} = -a^2(t)S^2(\chi)\sin^2\theta$$

(As presented many times...)

Covariant derivative

In Cartesian coordinate system (x,y,z) there are base unit vectors along coordinate lines: (1,0,0), (0,1,0), (0,0,1) which are the same at any point. Any vector can be expressed as a sum of its components*base_vectors. The derivative of a vector V (vector field V) is just the sum of derivatives_of_components*base_vectors:

$$\nabla_{\vec{U}}\vec{V} = U^j \nabla_j (V^i \vec{e}_i) = U^j V^i_{,j} \vec{e}_i \quad \text{and in this case:} \quad \nabla_j \vec{e}_i = 0$$

Now we use some base vectors which are not necessary unit vectors. One can choose:

$$dl^2 = \vec{e}_i \cdot \vec{e}_j \; dx^i dx^j \quad \Leftrightarrow \quad g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

Where the coordinates are not Cartesian and g_ij is the metric tensor. (One may ask if this decomposition is always possible. For our metric it is and for some other used in astrophysics also.)

Covariant derivative

As a 2D example we use polar coordinates on the plane. We define base vectors in agreement with the requirement and use their Cartesian components in calculations:

$$
\begin{aligned}
\vec{e}_r &= (\cos\phi, \sin\phi) \qquad \vec{e}_\phi = (-r\sin\phi, r\cos\phi) \qquad dl^2 = \vec{e}_r\cdot\vec{e}_r\ dr^2 + \vec{e}_\phi\cdot\vec{e}_\phi\ d\phi^2 \\
\nabla_r\vec{e}_r &= 0 \qquad \nabla_r\vec{e}_\phi = (-\sin\phi, \cos\phi) = \frac{1}{r}\vec{e}_\phi \\
\nabla_\phi\vec{e}_r &= (-\sin\phi, \cos\phi) = \frac{1}{r}\vec{e}_\phi \qquad \nabla_\phi\vec{e}_\phi = (-r\cos\phi, -r\sin\phi) = -r\vec{e}_r \\
\nabla_a\vec{e}_b &\equiv \Gamma^k_{ab}\vec{e}_k \qquad \Gamma_{ab|c} = \Gamma^k_{ab}g_{ck} \qquad \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\phi r} = \frac{1}{r} \qquad \Gamma^r_{\phi\phi} = -r \qquad \text{other vanish}
\end{aligned}
$$

The last row contains the definition of Christoffel symbols, which serve as coefficients when the derivative of a base vector is written down as a sum of other. We have found the symmetry of the symbols in our example; in fact this is their general property.

Covariant derivative

Components of the metric tensor are scalar products of base vectors pairs. Calculating derivatives as derivatives of a product and identifying product of base vectors in the results one gets the following:

$$
\begin{aligned}
\nabla_a g_{bc} &= g_{bc,a} = \Gamma^k_{ab}\vec{e}_k\vec{e}_c + \Gamma^k_{ac}\vec{e}_k\vec{e}_b = \Gamma^k_{ab}g_{kc} + \Gamma^k_{ac}g_{kb} \quad (+)\\
\nabla_b g_{ac} &= g_{ac,b} = \Gamma^k_{ab}\vec{e}_k\vec{e}_c + \Gamma^k_{bc}\vec{e}_k\vec{e}_a = \Gamma^k_{ab}g_{kc} + \Gamma^k_{bc}g_{ka} \quad (+)\\
\nabla_c g_{ab} &= g_{ab,c} = \Gamma^k_{ac}\vec{e}_k\vec{e}_b + \Gamma^k_{bc}\vec{e}_k\vec{e}_a = \Gamma^k_{ac}g_{kb} + \Gamma^k_{bc}g_{ka} \quad (-)\\
\frac{1}{2}(g_{bc,a} + g_{ac,b} - g_{ab,c}) &= \Gamma^k_{ab}\, g_{kc} \quad \Rightarrow \quad \Gamma^c_{ab} = \frac{1}{2}(g_{bk,a} + g_{ak,b} - g_{ab,k})g^{ck}
\end{aligned}
$$

If one knows the metric, he/she can calculate Christoffel symbols.

Back to our 2D example:

$$
\begin{aligned}
\vec{v} &= \dot{r}\vec{e}_r + \dot{\phi}\vec{e}\phi = \omega\vec{e}_\phi\\
\frac{d\vec{v}}{dt} &= \left(\dot{r}\nabla_r + \dot{\phi}\nabla_\phi\right)\vec{v} = \omega\nabla_\phi(\omega\vec{e}_\phi) = \omega^2\Gamma^r_{\phi\phi}\vec{e}_r = -\omega^2 r\ \vec{e}_r
\end{aligned}
$$

We obtain the standard result for centripetal acceleration; the method of calculations works.

Covariant derivative

Components of the derivative of a vector in a given coordinate system are also of interest:

$$
\begin{aligned}
g^{ak}\vec{e}_k\ \nabla_b\left(A^j\vec{e}_j\right) &= g^{ak}\vec{e}_k\ \left(A^j_{,b}\vec{e}_j + A^j\Gamma^l_{bj}\vec{e}_l\right) = g^{ak}g_{kj}A^j_{,b} + g^{ak}g_{kl}A^j\Gamma^l_{bj} \\
&= \delta^a_j A^j_{,b} + \delta^a_l A^j\Gamma^l_{bj} = A^a_{,b} + \Gamma^a_{bc}A^c
\end{aligned}
$$

Or in short:

$$
\nabla_b A^a = A^a_{,b} + \Gamma^a_{bc}A^c \qquad \nabla_b(A^jA_j) = A_jA^j,b + A^jA_{j,b} \quad \Rightarrow \quad \nabla_b A_a = A_{a,b} - \Gamma^c_{ab}A_c
$$

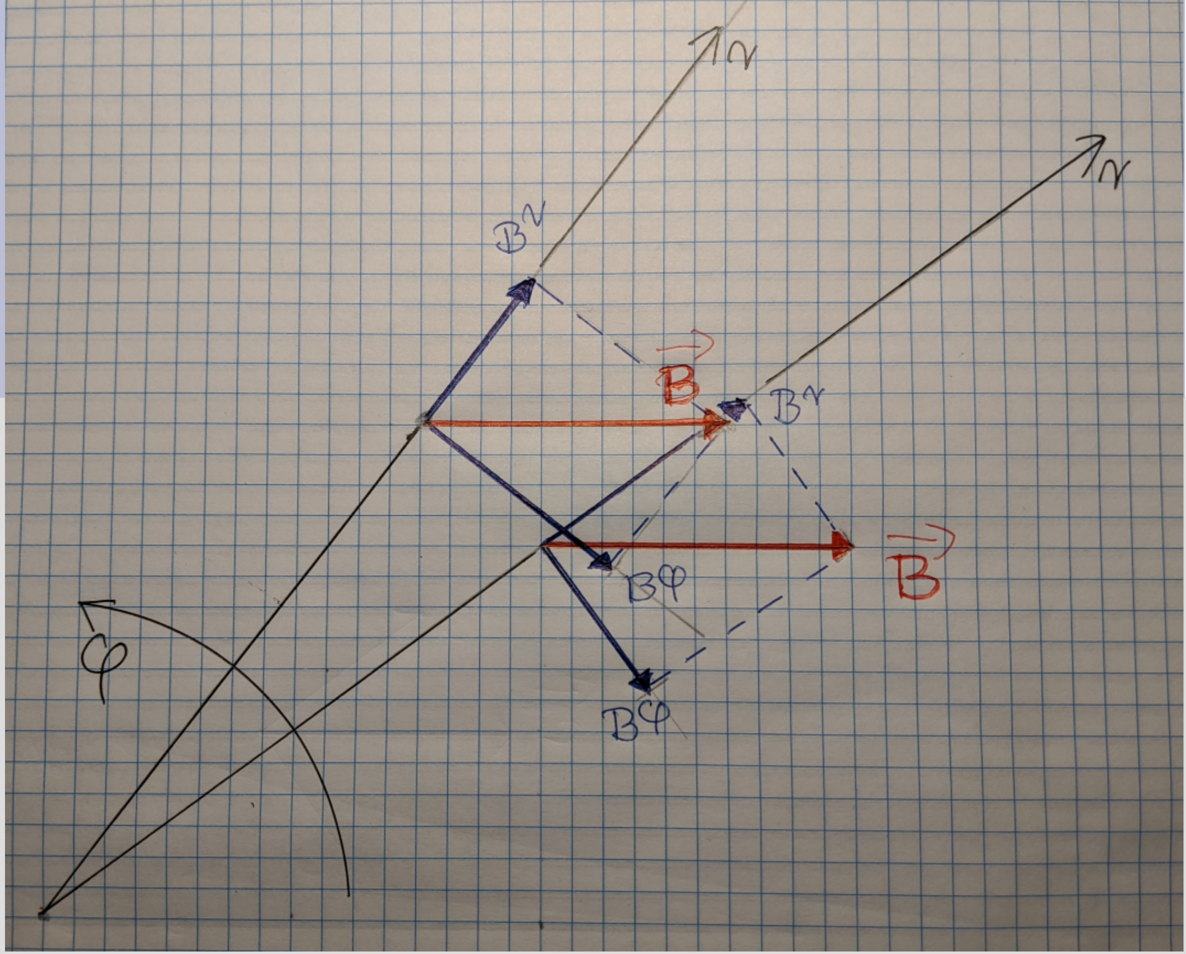
Tensors behave like vector products:

$$
\nabla_c\, T^{ab} = T^{ab}_{,c} + \Gamma^a_{cj}T^{jb} + \Gamma^b_{cj}T^{aj}
$$

So the method of calculating covariant derivatives applies to any vector or tensor field.

Covariant derivative

Another example: a uniform vector field in polar coordinates



$$B^x = B \qquad B^y = 0$$

$$B^r = \frac{\partial r}{\partial x} B^x + \frac{\partial r}{\partial y} B^y = B\cos\phi$$

$$B^\phi = \frac{\partial \phi}{\partial x} B^x + \frac{\partial \phi}{\partial y} B^y = -B\frac{\sin\phi}{r}$$

$$\begin{aligned}
\nabla_r B^r &= B^r_{,r} + \Gamma^r{}_{r\alpha} B^\alpha = 0 + 0 = 0 \\
\nabla_r B^\phi &= B^\phi_{,r} + \Gamma^\phi{}_{r\alpha} B^\alpha = +B\frac{\sin\phi}{r^2} + \Gamma^\phi{}_{r\alpha} B^\alpha = +B\frac{\sin\phi}{r^2} - B\frac{\sin\phi}{r^2} = 0 \\
\nabla_\phi B^r &= B^r_{,\phi} + \Gamma^r{}_{\phi\alpha} B^\alpha = -B\sin\phi - r\left(-B\frac{\sin\phi}{r}\right) = 0 \\
\nabla_\phi B^\phi &= B^\phi_{,\phi} + \Gamma^\phi{}_{\phi\alpha} B^\alpha = -B\frac{\cos\phi}{r} + \frac{1}{r} B\cos\phi = 0
\end{aligned}$$

Geodesic line

A line in any coordinate system can be parametrized as $x^a(\lambda)$
If the parameter is the length measured along the curve:

$$(d\lambda^2 = g_{ab} dx^a dx^b)$$

One can define a unit tangent vector as:

$$\vec{n} = \frac{dx^a}{d\lambda}\, \vec{e}_a \qquad n^a = \frac{dx^a}{d\lambda} \qquad \vec{n}\cdot\vec{n} = g_{ab}\frac{dx^a}{d\lambda}\frac{dx^b}{d\lambda} = 1$$

Now we require, that the vector tangent to the curve in a point does not change (remains parallel) when considering another point nearby:

$$\begin{aligned}
\frac{d}{d\lambda} &= \frac{dx^a}{d\lambda}\nabla_a \equiv n^a \nabla_a \\
0 &= n^b \nabla_b n^a = \frac{dx^b}{d\lambda}\left(\partial_b \frac{dx^a}{d\lambda} + \Gamma^a_{bc}\frac{dx^c}{d\lambda}\right) = \frac{dx^b}{d\lambda}\partial_b\left(\frac{dx^a}{d\lambda}\right) + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} \\
0 &= \frac{d^2x^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda}
\end{aligned}$$

This is the geodesic equation. It generalizes the concept of a straight line from a flat space (tangent vectors to a straight line at any point are parallel to each other)

Riemann tensor

The order of covariant derivatives may affect the result. Formal calculation using the definition of the derivative gives the following. The first pair of derivatives is treated explicitly, the result for the second is obtained by c←→ d exchange:

$$
\begin{aligned}
\nabla_c \nabla_d A^a &= \nabla_c(\nabla_d A^a) = \partial_c(\nabla_d A^a) + \Gamma^a_{cj}(\nabla_d A^j) - \Gamma^j_{cd}(\nabla_j A^a) \\
&= A^a_{,dc} + \Gamma^a_{dj,c} A^j + \Gamma^a_{dj} A^j_{,c} + \Gamma^a_{cj}(A^j_{,d} + \Gamma^j_{dk} A^k) - \Gamma^j_{cd}(\nabla_j A^a) \\
&= A^a_{,dc} + \Gamma^a_{dj,c} A^j + \Gamma^a_{dj} A^j_{,c} + \Gamma^a_{cj} A^j_{,d} + \Gamma^a_{cj}\Gamma^j_{dk} A^k - \Gamma^j_{cd}(\nabla_j A^a) \quad |+ \\
\nabla_d \nabla_c A^a &= \nabla_d(\nabla_c A^a) \\
&= A^a_{,cd} + \Gamma^a_{cj,d} A^j + \Gamma^a_{cj} A^j_{,d} + \Gamma^a_{dj} A^j_{,c} + \Gamma^a_{dj}\Gamma^j_{ck} A^k - \Gamma^j_{dc}(\nabla_j A^a) \quad |-
\end{aligned}
$$

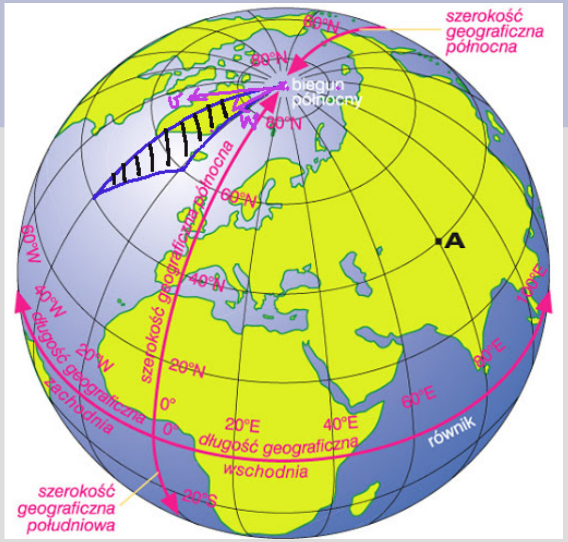
Finally one gets:

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) A^a = \left(\Gamma^a_{db,c} - \Gamma^a_{cb,d} + \Gamma^a_{cj}\Gamma^j_{db} - \Gamma^a_{dj}\Gamma^j_{cb}\right) A^b \equiv R^a_{bcd} A^b$$

Which is a formal definition of the Riemann tensor.

Parallel transport

For some geometric intuition we consider the so called parallel transport of a vector around a geodesic (i.e. having sides which are sections of geodesic lines) closed contour. “Parallel” means that its derivative along the path vanishes:



$$
\begin{aligned}
\frac{dx^c}{d\lambda}\nabla_c A^a &= 0 \;\Rightarrow\; \frac{dx^c}{d\lambda}A^a_{,c} = -\Gamma^a_{cb}A^b\frac{dx^c}{d\lambda} \;\Rightarrow \\
\Delta A^a &= \oint_{\partial S} A^a_{,c}dx^c = -\oint_{\partial S} \Gamma^a_{cb}A^b dx^c \\
&= -\int_S \left((\Gamma^a_{cb}A^b),d - (\Gamma^a_{db}A^b),c\right) dx^c dx^d
\end{aligned}
$$

where we have used the Stokes theorem. Expressing partial derivatives of A^b under the integral by expressions with Christoffel symbols one gets:

$$
\Delta A^a = -\int_S \left(\Gamma^a_{cb,d} - \Gamma^a_{db,c} - \Gamma^a_{cj}\Gamma^j_{db} + \Gamma^a_{cj}\Gamma^j_{db}\right) A^b dx^c dx^d = +\int_S R^a_{bcd}A^b dx^c dx^d
$$

Using an infinitesimally small triangle with vectors v, w tangent to its sides at the same vertex and proportional to the lengths of respective sides one gets

$$
\Delta A^a = R^a{}_{bcd}A^b v^c w^d
$$

A footnote: where are the tangent vectors? NOT in the curved space (space-time) but in a tangent flat space (at each point of the curved space there is a separate tangent space). GR: at any point of the space-time there should be a local tangent space with Minkowski metric. We cannot require that vectors in two distant points can be compared, but locally it is possible, two infinitesimally close locations may use the same tangent space.

Riemann tensor

Some algebraic properties of the Riemann tensor, which result directly from its formal definition and can be checked by calculations:

$$R_{abij} = +R_{ijab} = -R_{baij} = -R_{abji} = +R_{baji}$$

$$R_{aijk} + R_{ajki} + R_{akij} = 0$$

In 4D space time Riemann tensor has 20 independent components (not 4^4=256 as in general case).
Checking the definition gives also the Bianchi identity:

$$\nabla_i R^a{}_{bjk} + \nabla_j R^a{}_{bki} + \nabla_k R^a{}_{bij} = 0$$

Algebraic properties of the Riemann tensor lead to some classification of possible space-times. We are not following this topic.

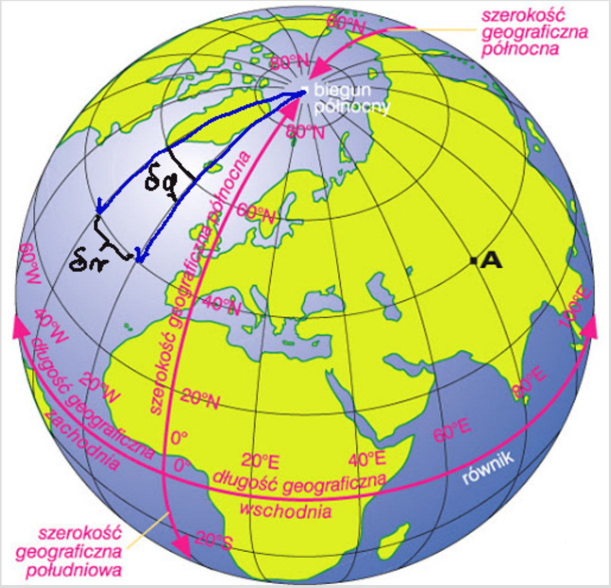
The space is flat only if the Riemann tensor vanishes everywhere.

Geodesic deviation

What is the second time derivative of the distance between two point masses,both moving on geodesic lines close to each other ?

A Galilean kinematics example: two test particles start from the pole in t=0 and move with a constant velocity v along two meridians separated by \delta\phi:

$$\begin{aligned} \delta r &= R\sin\theta\delta\phi \qquad \theta = \frac{vt}{R} \\ \frac{d\delta r}{dt} &= v\cos\theta\delta\phi \\ \frac{d^2\delta r}{dt^2} &= -\frac{v^2}{R}\sin\theta\delta\phi \end{aligned}$$



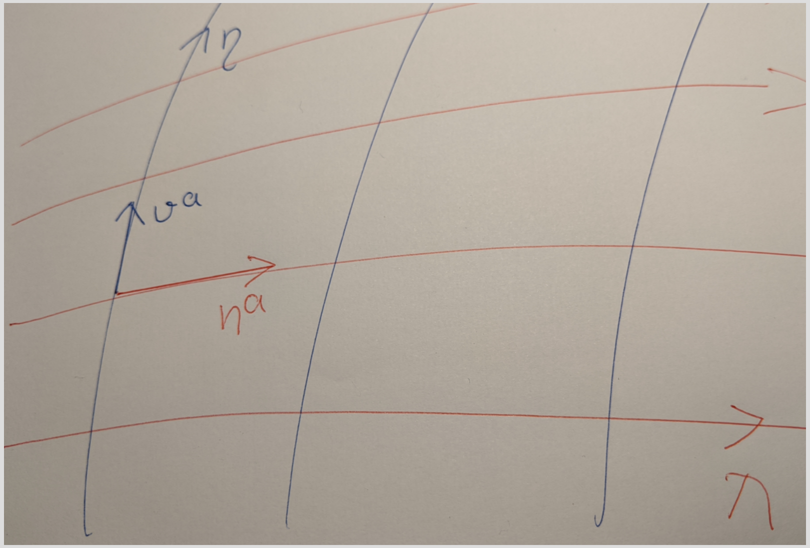
R is the radius of the sphere, \theta angle from the pole \delta r separation of the two points. 3D perspective: centripetal acceleration projected onto the surface. 2D perspective: relative acceleration caused by the curvature of space.

Newtonian dynamics: tidal forces?

Geodesic deviation

GR: no gravity which is hidden in the geometry of space-time. We look for analogy.
We consider 2D surface “made of geodesics” , parametrized by lambda (which goes along geodsic lines) and eta, which is constant along any particular geodesic. The surface consists of the points $x^a(\lambda,\eta)$
Vector n is tangent to geodesic, v measures the distance between them

$$n^a = \frac{\partial x^a}{\partial \lambda} \qquad v^a = \frac{\partial x^a}{\partial \eta}$$



Simple identity which we use also in the final calculation:

$$\underline{n^i \nabla_i v^a} = \frac{\partial^2 x^a}{\partial \eta \partial \lambda} + \Gamma^a_{ij} n^i v^j = \frac{\partial^2 x^a}{\partial \lambda \partial \eta} + \Gamma^a_{ij} v^i n^j = \underline{v^i \nabla_i n^a}$$

$$\begin{aligned}
n^i \nabla_i n^j \nabla_j v^a &= n^i \nabla_i (\underline{n^j \nabla_j v^a}) = n^i \nabla_i (\underline{v^j \nabla_j n^a}) \\
&= \underline{n^i \nabla_i (v^j)} \nabla_j n^a + n^i v^j \nabla_i \nabla_j n^a \\
&= \underline{v^i \nabla_i (n^j)} \nabla_j n^a + n^i v^j \nabla_i \nabla_j n^a \\
&= v^i \nabla_i (\underbrace{n^j \nabla_j n^a}_{0}) - v^i n^j \nabla_i \nabla_j n^a + n^i v^j \nabla_i \nabla_j n^a \\
&= 0 + v^i n^j (\nabla_i \nabla_j - \nabla_j \nabla_i) n^a
\end{aligned}$$

$$n^i \nabla_i n^j \nabla_j v^a = R^a_{bij} n^b v^i n^j$$

The final result gives another intuition related to the Riemann tensor:
in GR it is somehow related to tidal forces.

Geodesic deviation

Application: assuming all velocities vanish in the comoving coordinates in the uniform Universe, we assume velocities caused by perturbations to be low, so n=u~(1,0,0,0) – four-velocity. Substituting:

$$g^a = R^a{}_{bij}\delta^b_0 v^i \delta^j_0 = R^a{}_{0i0} v^i$$

In connection with interpretation of the result as an action of tidal forces we name it “relative accelaration”. For small relative distances $v^i = \Delta x^\alpha \delta^i_\alpha$
One gets:

$$\Delta g^\chi = R^\chi{}_{0\chi 0}\Delta\chi \qquad \Delta g^\theta = R^\theta{}_{0\theta 0}\Delta\theta \qquad \Delta g^\phi = R^\phi{}_{0\phi 0}\Delta\phi$$

The divergence of the acceleration calculated with finite differences reads:

$$\begin{aligned}\sum \frac{\Delta g^\alpha}{\Delta x^\alpha} &= R^\chi{}_{0\chi 0} + R^\theta{}_{0\theta 0} + R^\phi{}_{0\phi 0} \equiv R_{00} \quad \leftarrow \text{Ricci tensor component} \\ &= -3\frac{\ddot{a}}{a} \quad \text{in our metric} \qquad \text{(See below)} \\ &= +\frac{4\pi G}{c^2}(\epsilon + 3P) \quad \text{our metric and Einstein equations}\end{aligned}$$

This is in a sense equivalent to Poisson equation of Newtonian gravity

Einstein equations

We define the Ricci tensor and the scalar of curvature:

$$R_{ab} = R^{i}{}_{aib} \qquad R_{ab} = R_{ba} \qquad R = g^{ab} R_{ab}$$

(summation convention). The symmetry of the Ricci tensor follows the symmetry of the Riemann tensor. Introducing action S which consists of the gravitational and material parts:

$$S = S_g + S_m = \int \left(\frac{c^4}{16\pi G} R + \mathcal{L}_m \right) \sqrt{-g} d_4 x$$

where material part is represented by the Lagrangian density and using variation principle one obtains the Einstein equations [EE]:

$$\delta S = 0 \longrightarrow R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} T_{ab}$$

(It is not so easy, but doable.)

Einstein equations

There is also a “more general” form of the EE including cosmological term:

$$R^a_b - \frac{1}{2}R\delta^a_b = \frac{8\pi G}{c^4}T^a_b + \Lambda\delta^a_b$$

which preserves the invariance of the equations and does not influence the local measurements if the cosmological constant Lambda is small enough.

RHS: the energy-momentum tensor usually takes the form:

$$T^a_{\ b} = (\epsilon + P)u^a u_b - P\delta^a_{\ b}$$

which represents the ideal fluid. No energy transport (except advection: carrying thermal energy with the matter) is present, neither friction, internal stress, etc which would require separate terms. In a comoving coordinate system one has:

$$u^a = \delta^a_{\ 0} \longrightarrow T^0_{\ 0} = \epsilon,\ T^\chi_{\ \chi} = T^\theta_{\ \theta} = T^\phi_{\ \phi} = -P$$

This is the case of our coordinates as well. (In general the comoving system may not exist e.g. when the matter is rotating differentially.)

Universe model: the details

Using our coordinates (but replacing ct → x^0 to get rid of c):

$$ds^2 = (dx^0)^2 - a^2(x^0)\left(d\chi^2 + S^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)$$

$$\vec{e}_0 = (1,0,0,0) \quad \vec{e}_\chi = (0,a,0,0) \quad \vec{e}_\theta = (0,0,aS(\chi),0) \quad \vec{e}_\phi = (0,0,0,aS(\chi)\sin\theta)$$

we get for Christoffel symbols and the Riemann tensor components:

$$\begin{aligned}
\Gamma^0_{\chi\chi} &= a\dot{a} \quad \Gamma^\chi_{0\chi} = \frac{\dot{a}}{a} \\
\Gamma^0_{\theta\theta} &= a\dot{a}S^2 \quad \Gamma^\theta_{0\theta} = \frac{\dot{a}}{a} \quad \Gamma^\chi_{\theta\theta} = -SC \quad \Gamma^\theta_{\chi\theta} = \frac{C}{S} \\
\Gamma^0_{\phi\phi} &= a\dot{a}S^2\sin^2\theta \quad \Gamma^\phi_{0\phi} = \frac{\dot{a}}{a} \quad \Gamma^\chi_{\phi\phi} = -SC\sin^2\theta \\
\Gamma^\phi_{\chi\phi} &= \frac{C}{S} \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \quad \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

$$\begin{aligned}
R^\chi{}_{0\chi0} &= R^\theta{}_{0\theta0} = R^\phi{}_{0\phi0} = -\frac{\ddot{a}}{a} \quad R_{00} = -3\frac{\ddot{a}}{a} \quad R^0{}_0 = -3\frac{\ddot{a}}{a} \\
R^0{}_{\chi0\chi} &= a\ddot{a} \quad R^\theta{}_{\chi\theta\chi} = \dot{a}^2 + k^2 \quad R^\phi{}_{\chi\phi\chi} = \dot{a}^2 + k^2 \\
R_{\chi\chi} &= R^i{}_{\chi i\chi} = a\ddot{a} + 2\dot{a}^2 + 2k^2 \quad R^\chi{}_\chi = -\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a} - 2\frac{k^2}{a^2} \\
R^\theta{}_\theta &= R^\phi{}_\phi = R^\chi{}_\chi \\
R &= R^i{}_i = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k^2}{a^2}\right)
\end{aligned}$$

which defines also the Ricci tensor and the curvature scalar.

Universe model: the details

The LHS of the EE contains only two different components

$$R^0{}_0 - \frac{1}{2}R\delta^0_0 = 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2}$$
$$R^\chi{}_\chi - \frac{1}{2}R\delta^\chi_\chi = 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k^2}{a^2}$$

(The second eq has another two copies on the diagonal and all off-diagonal LHS of EE vanish). Substituting energy-momentum tensor into the RHS we have:

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k^2c^2}{a^2} = +\frac{8\pi G}{c^2}\epsilon$$
$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k^2c^2}{a^2} = -\frac{8\pi G}{c^2}P$$

$$\longrightarrow$$

$$\frac{\dot{a}^2}{a^2} + \frac{k^2c^2}{a^2} = +\frac{8\pi G}{3c^2}\epsilon$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P)$$

- the classic form of the EE in application to isotropic and uniform Universe models. The acceleration equation shows that the internal pressure is also a source of gravity which tends to slow down the expansion.

Universe model: the details

The version including cosmological term reads:

$$\frac{\dot{a}^2}{a^2} + \frac{k^2c^2}{a^2} = +\frac{8\pi G}{3c^2}\epsilon + \frac{1}{3}\Lambda c^2$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{1}{3}\Lambda c^2$$

$$\epsilon_\Lambda = +\frac{c^4}{8\pi G}\Lambda$$

$$P_\Lambda = -\frac{c^4}{8\pi G}\Lambda$$

We formally define variables of the dimension of energy density and pressure proportional to the cosmological constant and substitute them into eqs:

$$\frac{\dot{a}^2}{a^2} + \frac{k^2c^2}{a^2} = +\frac{8\pi G}{3c^2}(\epsilon_M + \epsilon_\Lambda)$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon_M + 3P_M + \epsilon_\Lambda + 3P_\Lambda)$$
$$= -\frac{4\pi G}{3c^4}(\epsilon_M + 3P_M - 2\epsilon_\Lambda)$$

Now the cosmological term plays the role of another matter component with an odd equation of state (which can be generalized by assuming that the pressure to density ratio factor can have other values than w=-1 as in the case of cosmological constant). This component is called dark energy. Today the ordinary matter pressure vanishes. For w=-1 and dark energy density exceeding half of the ordinary matter energy density one has accelerated expansion.

History: the static Universe of Einstein

At the time (circa 1915) the expansion of the Universe was not discovered yet. The static model was plausible and this was (they say) the reason Einstein introduced the cosmological constant to his equations. Static means that time derivatives vanish automatically which gives:

$$\begin{aligned} \frac{kc^2}{a^2} &= +\frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 \\ 0 &= -\frac{4\pi G}{3}\rho + \frac{1}{3}\Lambda c^2 \end{aligned}$$

(2 equations 4 unknowns). Eliminating rho:

$$\frac{kc^2}{a^2} = \Lambda c^2 \quad \Rightarrow \quad k = +1 \quad \Lambda = \frac{1}{a^2}$$

→ static model is closed (k=+1, 3D space is a 3D sphere), Lambda and the radius of curvature are simply related. For a=c/H_0 the density is critical/1.5

Weak field limit

We are checking the nonrelativistic limit of the Einstein Equations.
Metric: Minkowski + 1st order. Velocity <<c.

$$u^a = \left(1, \frac{\delta\vec{v}}{c}\right) \qquad \frac{|\delta\vec{v}|}{c} \ll 1$$

$$\frac{d}{dct}\left(\frac{\delta v^\alpha}{c}\right) + \Gamma^\alpha{}_{bc} u^b u^c = 0$$

All Christoffel symbols are of the first or higher order so only u^0 substituted above may give 1st order terms →

$$\Gamma^\alpha{}_{00} \approx g^{\alpha\alpha}\Gamma_{00|\alpha} = (-1) * \left(-\frac{1}{2} g_{00,\alpha}\right)$$

$$\frac{1}{c^2}\frac{d\delta v^\alpha}{dt} + \frac{1}{2} g_{00,\alpha} = 0$$

$$\frac{d\delta v^\alpha}{dt} = -\nabla_\alpha \delta\phi \ \Rightarrow \quad g_{00} = 1 + 2\frac{\delta\phi}{c^2}$$

Comparing the geodesic equation with the Newtonian equation of motion we get the form of one metric component in the weak field limit. We use “delta phi” for the potential having in mind its small perturbations depending on local density perturbations. The Newtonian potential of positive on average matter density in infinite space is impossible to define properly.

Weak field limit

Postulating

$$g_{00} = 1 + 2\delta\psi \qquad g_{xx} = g_{yy} = g_{zz} = -1 + 2\delta\psi$$

one gets Christoffel symbols, Riemann and Ricci tensors and finally:

$$R^0{}_0 - \frac{1}{2}R\delta^0{}_0 = +2\triangle\delta\psi \qquad R^\alpha{}_\alpha - \frac{1}{2}R\delta^\alpha{}_\alpha = -2\delta\ddot{\psi}$$

$$T^0{}_0 = \delta\rho c^2 \qquad T^\alpha{}_\alpha = 0$$

$$\triangle\delta\phi = 4\pi G\delta\rho \qquad \delta\ddot{\phi} = 0$$

Which is the Poisson equation for potential perturbations caused by matter density perturbations. Another equation says that the time evolution of the potential is slow. (Since matter moves slowly, changes in its distribution are slow. The spatial gradients of the potential are larger than its time derivative.)

(Only the distribution of matter at any given moment defines the potential.)
The LHS of the equations is valid as long as |potential|<<c^2. It does not require that density perturbations are small.
Weak field equations are linear.

Weak field limit

The assumption of slow motion is necessary for our equations to be valid (otherwise potential time derivatives may not of higher order). Assuming that we use coordinates comoving with the unperturbed matter, we may assume that the peculiar motions are small perturbations, so kinematics is non-relativistic. But it does not imply the non-relativistic equation of state. The off-diagonal components in energy-momentum tensor are of the higher order (as including spatial components of 4-velocity) so:

$$
\begin{aligned}
R_{00} &= \frac{8\pi G}{c^4}\left(T_{00} - \frac{1}{2} T g_{00}\right) \\
T &= T^0_{\ 0} + T^x_{\ x} + T^y_{\ y} + T^z_{\ z} = \delta\epsilon - 3\delta P \\
R_{00} &= \frac{8\pi G}{c^4}\left(\delta\epsilon - \frac{1}{2}(\delta\epsilon - 3\delta P)\right) \\
\triangle\phi &= -\nabla \cdot \delta\vec{g} = \frac{4\pi G}{c^2}(\delta\epsilon + 3\delta P)
\end{aligned}
$$

Which is a generalization of the Poisson equation. “delta g” is “-gradient of the potential” here, but its relation to matter perturbations is the same as implied by geodesic deviation equation.

Hydrodynamics

If the matter moves in a coordinate system its time evolution can be properly followed by the “matter derivative”:

$$\frac{D}{dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

(It is not enough to know changes of any matter parameter in a point of space, it is also necessary to take into account the change of the position of the matter element in question).

The equations of the classic (non-relativistic) hydrodynamics read:

$$\begin{aligned} \frac{D\vec{v}}{dt} &= -\frac{\nabla P}{\rho} - \nabla\Phi && \text{(Euler)} \\ \frac{D\rho}{dt} &= -\rho\nabla \cdot \vec{v} && \text{(continuity)} \\ \nabla^2\Phi &= 4\pi G\rho && \text{(Poisson)} \\ P &= P(\rho) && \text{(eq. of state)} \end{aligned}$$

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \equiv \rho_0(1+\delta) \\ \vec{v} &= \vec{v}_0 + \delta\vec{v} \equiv \frac{\dot{a}}{a}\vec{r} + \delta\vec{v} \\ \Phi &= \Phi_0 + \delta\Phi \\ P &= P_0 + \delta P \equiv P_0 + c_S^2\delta\rho \equiv P_0 + \rho_0 c_S^2\delta \end{aligned}$$

The variables are expanded to the 1st order. The velocity is given as the uniform expansion part plus the peculiar motion. Pressure perturbations are adiabatic since all energy transport phenomena are neglected.

Hydrodynamics

After linearizing equations we see that matter derivative simplifies to:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} \approx \frac{\partial}{\partial t} + \frac{\dot{a}}{a}\vec{r} \cdot \nabla_{\vec{r}} = \frac{d}{dt}$$

The continuity equation gives:

$$\left(\frac{\partial}{\partial t} + \frac{\dot{a}}{a} r^j \nabla_j\right) \delta\rho = -\,\delta\rho \nabla \left(\frac{\dot{a}}{a}\vec{r}\right) - \rho_0 \nabla \delta\vec{v}$$

$$\frac{1}{\rho_0}\frac{d}{dt}\delta\rho = -\,3\frac{\dot{a}}{a}\frac{\delta\rho}{\rho_0} - \nabla\delta\vec{v}$$

$$\frac{1}{\rho_0}\frac{d}{dt}(\rho_0\delta) + 3\frac{\dot{a}}{a}\delta = -\,\nabla\delta\vec{v}$$

$$\frac{d}{dt}\delta = -\nabla\delta\vec{v}$$

Euler eq., its divergence:

$$\left(\frac{\partial}{\partial t} + \frac{\dot{a}}{a} r^j \nabla_j\right) \delta v^i + \frac{\dot{a}}{a}\delta v^i = -\frac{\nabla_i \delta P}{\rho_0} - \nabla_i \delta\Phi$$

$$\frac{d}{dt}(\nabla\delta\vec{v}) + 2\frac{\dot{a}}{a}\nabla\delta\vec{v} = -\,c_S^2\nabla^2\delta - \nabla^2\delta\Phi$$

$$\ddot{\delta} + 2\frac{a}{a}\dot{\delta} = +\,c_S^2\nabla^2\delta + 4\pi G\rho_0\delta$$

And finally, for a single plane wave we get the equation describing the evolution of small relative density perturbations. The are written down in a coordinate system comoving with the unperturbed Universe.

$$\delta = \delta_{\vec{k}}\ \exp(i\vec{k}\vec{x}) \qquad \vec{r} = a\vec{x} \qquad \lambda = \frac{2\pi a}{|\vec{k}|}$$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + \left(\frac{k^2 c_S^2}{a^2} - 4\pi G\rho_0\right)\delta = 0$$

Relativistic hydrodynamics

Bianchi identity implies vanishing of the LHS of the EE → $\nabla_b T^{ab} = 0$
Calculating the divergence we get:

$$D/Ds \equiv u^b \nabla_b .$$

$$T^{ab} = (\epsilon + P) u^a u^b - P g^{ab} \qquad \nabla_b T^{ab} = 0$$

$$u^a \frac{D}{Ds}(\epsilon + P) + u^a (\epsilon + P) \nabla_b u^b + (\epsilon + P) a^a - \nabla^a P = 0$$

“a” is the acceleration. Since 4-velocity is normalized its derivative (“a”) must be perpendicular and multiplying by 4-velocity we obtain the energy conservation equation:

$$\frac{D}{Ds}\epsilon + (\epsilon + P) \nabla_b u^b = 0$$

In a small region the Hubble expansion is non-relativistic, so locally matter derivative acting on perturbed variables does not include peculiar velocities and the energy equation becomes:

$$\frac{d}{dt}\delta\epsilon + 3\frac{\dot{a}}{a}(\delta\epsilon + \delta P) + (\epsilon + P)\nabla\delta\vec{v} = 0$$

We examine $\frac{d}{dt}\left(\frac{\delta\epsilon}{\epsilon + P}\right)$

finally obtaining

$$\frac{d}{dt}\left(\frac{\delta\epsilon}{\epsilon + P}\right) = -\nabla\delta\vec{v}$$

(Notes)

Relativistic hydrodynamics

Again we use the slow motion approximation replacing:

$$-\nabla_i \delta\Phi \longrightarrow \delta g_i \qquad \rho \longrightarrow (\epsilon + P)/c^2$$

And using eq.of motion in the form

$$\left(\frac{\partial}{\partial t} + \frac{\dot{a}}{a} r^j \nabla_j\right) \delta v^i + \frac{\dot{a}}{a} \delta v^i = -\frac{c^2 \nabla_i \delta P}{\epsilon + P} + \delta g^i$$

Its divergence:

$$\frac{d}{dt}(\nabla \delta \vec{v}) + 2\frac{\dot{a}}{a} \nabla \delta \vec{v} \;\; = \;\; -c_S^2 \nabla^2 \frac{\delta\epsilon}{\epsilon + P} - \frac{4\pi G}{c^2}(\delta\epsilon + 3\delta P)$$

$$\frac{d}{dt}\left(-\frac{d\delta}{dt}\right) + 2\frac{\dot{a}}{a}\left(-\frac{d\delta}{dt}\right) = -\, c_S^2 \nabla^2 \delta - 4\pi G \rho_0 \frac{\epsilon + P}{\epsilon}(1 + 3c_S^2/c^2)\delta$$

For a single plane wave:

Delta= $\dfrac{\delta\epsilon}{\epsilon + P}$

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + \left[\frac{k^2 c_S^2}{a^2} - 4\pi G \rho_0 (1 + w)(1 + 3c_S^2/c^2)\right]\delta = 0$$

Up to “relativistic corrections” this is the same perturbation equation as before.

Współrzędne ,,astronomiczne"

Nie mierzymy bezpośrednio czasu, w którym obserwowane fotony zostały wyemitowane.
Nie mierzymy czasu, jaki od tego momentu upłynął.
Nie mierzymy bezpośrednio odległości do źródła.
Mierzymy bezpośrednio przesunięcie ku czerwieni! ==>
uczyńmy je zmienną niezależną w równaniu ewolucji Wszechświata.

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3c^2}(\epsilon + \epsilon_\Lambda)$$

$$1 + z = \frac{a_0}{a(t)} \quad \Rightarrow \quad \frac{1}{a}\frac{da}{dt} = -\frac{1}{1+z}\frac{dz}{dt}$$

$$\frac{1}{(1+z)^2}\left(\frac{dz}{dt}\right)^2 + \frac{kc^2}{a_0^2}(1+z)^2 = \frac{8\pi G}{3c^2}(\epsilon + \epsilon_\Lambda)$$

Podstawiając do górnego równania ,,dzisiejsze" wartości mamy:

$$H_0^2 + \frac{kc^2}{a_0^2} = \frac{8\pi G}{3c^2}(\epsilon(t_0) + \epsilon_\Lambda(t_0))$$

Parametry gęstości

W wyróżnionym przypadku modelu płaskiego ($k = 0$) mamy

$$\epsilon_M(t_0) + \epsilon_\Lambda(t_0) = \frac{3H_0^2}{8\pi G}c^2 \equiv \rho_c c^2$$

gdzie ρ_c jest charakterystyczną, tzw krytyczną gęstością. Jeśli oznaczyć:

$$\epsilon_M(t_0) \equiv \Omega_M \rho_c c^2 \qquad \epsilon_\Lambda(t_0) \equiv \Omega_\Lambda \rho_c c^2$$

to po przekształceniach mamy:

$$1 + \frac{kc^2/H_0^2}{a_0^2} = \Omega_M + \Omega_\Lambda \quad \Leftrightarrow \quad \frac{kc^2}{a_0^2} = H_0^2(\Omega_M + \Omega_\Lambda - 1)$$

albo:

$$\Omega_K \equiv 1 - \Omega_M - \Omega_\Lambda \quad \Rightarrow \quad \frac{kc^2}{a_0^2} = -\Omega_K H_0^2$$

Zależność $t(z)$

Przy $z < 10^3$ można pominąć ciśnienie materii ($P_M \ll \epsilon_M$). Używając I zasady termodynamiki

$$\epsilon_M \propto a^{-3} \quad \Rightarrow \quad \epsilon_M(t) = \Omega_M \rho_c c^2 (1+z)^3$$

$$\epsilon_\Lambda \propto a^{0} \quad \Rightarrow \quad \epsilon_\Lambda(t) = \Omega_\Lambda \rho_c c^2$$

Podstawiając w równaniu na tempo ekspansji $a = a_0/(1+z)$ i powyższe mamy

$$\frac{1}{(1+z)^2}\left(\frac{dz}{dt}\right)^2 + \frac{kc^2}{a_0^2}(1+z)^2 = H_0^2\left(\Omega_M(1+z)^3 + \Omega_\Lambda\right)$$

a używając definicji Ω_K:

$$\frac{1}{(1+z)^2}\left(\frac{dz}{dt}\right)^2 = H_0^2\left(\Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right)$$

Ostatecznie otrzymujemy zależność (de facto znak jest przeciwny):

$$\frac{dt}{dz} = \frac{1}{H_0}\frac{1}{1+z}\frac{1}{\sqrt{\Omega_M(1+z)^3+\Omega_K(1+z)^2+\Omega_\Lambda}} \equiv \frac{1}{H_0}\frac{1}{1+z}\frac{1}{f(z)}$$

Wracając do równania opisującego tempo ekspansji

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2(\Omega_M(1+z)^3+\Omega_K(1+z)^2+\Omega_\Lambda)$$

Czyli $H(z) = H_0\ f(z)$ i dlatego zwykle uzywa sie $h(z) \equiv f(z)$ na oznaczenie tej zależności. Czas od z" (*look-back time*) to:

$$t_b(z) = \frac{1}{H_0}\int_0^z \frac{dz'}{(1+z')f(z')}$$

i jest skończony przy $z \to \infty$ ($a \to 0$) Wobec tego można mierzyć czas od *początku* do epoki z:

$$t(z) = \frac{1}{H_0}\int_z^\infty \frac{dz'}{(1+z')f(z')}$$

END

Zależność $\chi(z)$

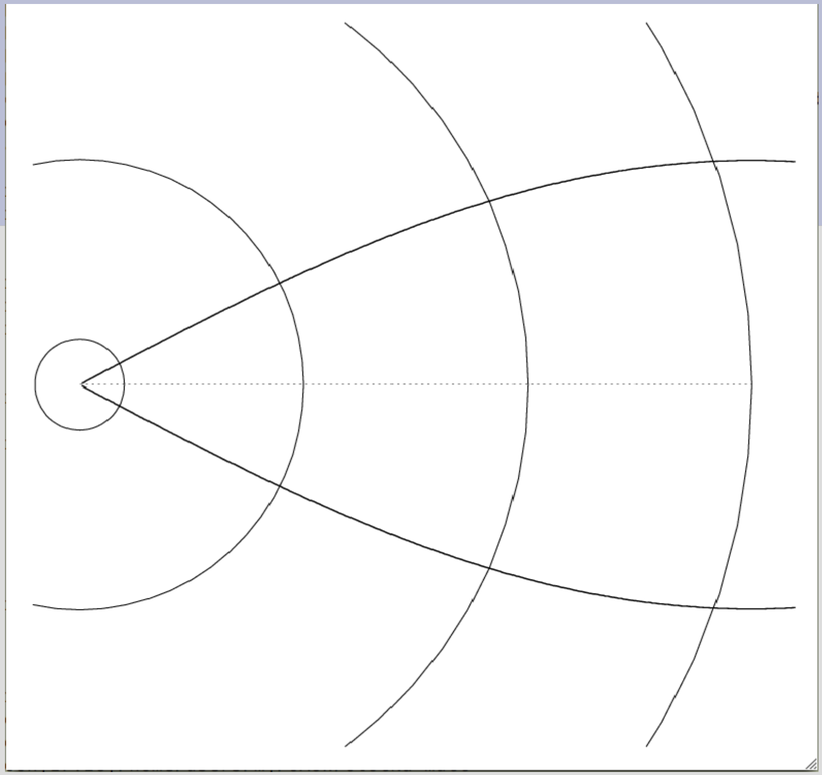
Poprzednio wyraziliśmy radialną współrzędną obiektu, który wysłał sygnał w czasie t_{em}, odebrany w czasie t_{obs} jako

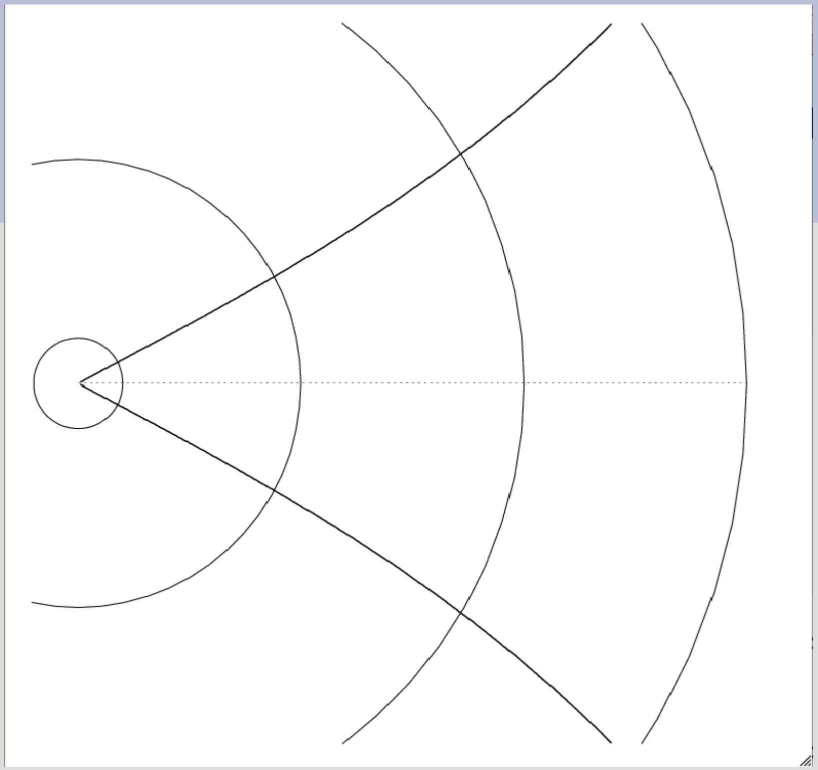
$$\chi = \int_{t_{em}}^{t_{obs}} \frac{cdt}{a(t)}$$

co było trudne do bezpośredniego wykorzystania, ale pozwalało znaleźć związek $z \leftrightarrow a(t)$. Używając $t(z)$ mamy:

$$\chi(z) = \int_0^z \frac{1+z'}{a_0}\frac{cdt}{dz'}dz' = \frac{c/H_0}{a_0}\int_0^z \frac{dz'}{f(z')}$$

gdzie prawa strona jest całką ze znanej funkcji.





Z takich fragmentów możemy skleić 2D sferę a z takich 2D przestrzeń o k=-1

Horyzont

Nawet obserwując źródła o bardzo dużym przesunięciu ku czerwieni, nie sięgamy do nieskończoności, ale do HORYZONTU o współrzędnej:

$$\chi_H = \frac{c/H_0}{a_0} \int_0^\infty \frac{dz}{f(z)} < \infty$$

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{dz}{(1+z)f(z)} < \infty$$

(Czas ewolucji też jest skończony.) Obserwowalny Wszechświat znajduje się wewnątrz sfery o promieniu \chi_H. ,,Dzisiejsza" objętość tego obszaru też jest skończona:

$$V_H = 4\pi a_0^3 \int_0^{\chi_H} S^2(\chi) d\chi < \infty$$

I mieści się w nim skończona liczba galaktyk, gwiazd, atomów,...